DYNAMICAL SYSTEMS OF SIMPLICES IN DIMENSION 2 OR 3

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Abstract. Let $\mathcal{T}_0 = (A_0^0 \cdots A_0^d)$ be a d-simplex, G_0 its centroid, $\mathcal{S}$ its circumsphere, O the center of $\mathcal{S}$. Let (A_1^i) be the points where $\mathcal{S}$ intersects the lines $(G_0 A_0^i)$, $\mathcal{T}_1$ the d-simplex $(A_1^0 \cdots A_1^d)$, and G_1 its centroid. By iterating this construction, a dynamical system of d-simplices $(\mathcal{T}_i)$ with centroids (G_i) is constructed. For $d = 2$ or 3, we prove that the sequence $(OG_i)_i$ is decreasing and tends to 0. We consider the sequences $(\mathcal{T}_{2i})_i$ and $(\mathcal{T}_{2i+1})_i$; for $d = 2$ they converge to two equilateral triangles with at least quadratic speed; for $d = 3$ they converge to two isosceles tetrahedra with at least geometric speed. In this last case, we give an explicit expression of the lengths of the edges of the limit form. We show also that if $\mathcal{T}_0$ is a planar cyclic quadrilateral then $(\mathcal{T}_n)$ converges to a rectangle with at least geometric speed or eventually to a square with a speed that is conjectured as cubic. The proofs are largely algebraic and use Gröbner basis computations.

1. Introduction

1.1. **The general problem.** Let $\mathcal{T}_0 = (A_0^0 \cdots A_0^d)$ be a d-simplex, G_0 its centroid, $\mathcal{S}$ its circumsphere in $\mathbb{R}^d$, O the center of $\mathcal{S}$. Let (A_1^i) be the points where $\mathcal{S}$ intersects the lines $(G_0 A_0^i)$ respectively, $\mathcal{T}_1$ the d-simplex $(A_1^0 \cdots A_1^d)$, and G_1 its centroid. If we iterate this construction then we produce a dynamical system of d-simplices $(\mathcal{T}_i)$ with centroids (G_i).

Let M, N be two points of $\mathbb{R}^d$; MN refers to the euclidean norm of the vector $\overrightarrow{MN}$.

Numerical investigations, using Maple, with thousands of random simplices in dimensions up to 20 indicate that:

i) the sequence $(OG_i)_i$ is decreasing and tends to 0 (from [6]).
ii) the sequences $(\mathcal{T}_{2i})_i$ and $(\mathcal{T}_{2i+1})_i$ converge to two d-simplices with centroid O.

Remark. *The condition "circumcenter=centroid" for a d-simplex is equivalent to the statement that the sum of the squares of the edge lengths of each facet is the same for all $d+1$ facets.*

We prove the assertions i) and ii) in the cases $d = 2$ and $d = 3$. We conjecture that the result is valid for any d.
From now on, we assume that the radius of $\mathcal{S}$ is 1.

1.2. **The case $d = 3$.** $\mathcal{T}_i = (A_i B_i C_i D_i)$ is a tetrahedron such that its four vertices are not coplanar. These vertices are ordered and $\mathcal{T}_i$ is viewed as a point of $\mathcal{S}^4$
Let ϕ and ψ be the functions which transform $\mathcal{T}_0$ into $\mathcal{T}_1$ and G_1.

Remark. *i) ϕ does not admit any non planar tetrahedron as fixed point.*
ii) If $\mathcal{T}$ is an isosceles tetrahedron then $\mathcal{T}$ is a fixed point of $\phi \circ \phi$.

Our main result is the following:

The sequences $(\mathcal{T}_{2i})_{i\in\mathbb{N}}$ and $(\mathcal{T}_{2i+1})_{i\in\mathbb{N}}$ are well defined and converge, with at least geometric speed, to two non planar isosceles tetrahedra that are symmetric with respect to O.

We consider also the degenerate case where $\mathcal{T}_0$ is a planar cyclic convex quadrilateral; then the sequence $(\mathcal{T}_i)_i$ converges, with at least geometric speed, to a rectangle. If $\mathcal{T}_0$ is a harmonic quadrilateral then the limit form is a square and we conjecture that the convergence is with order three.

Moreover, in both cases, we give explicit expressions for the lengths of the edges of the limit form from the ones of $\mathcal{T}_0$.

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1.3. **The case** $d = 2$**.** $\mathcal{T}_i = (A_iB_iC_i)$ is a triangle such that its vertices are pairwise distinct.

We prove that the sequences $(\mathcal{T}_{2i})_{i\in\mathbb{N}}$ and $(\mathcal{T}_{2i+1})_{i\in\mathbb{N}}$ are well defined and converge to two equilateral triangles that are symmetric with respect to O. Moreover, these sequences converge with at least quadratic speed.

Thus if the dimension is two, it can be observed a much more quick convergence that in the dimension three case.

1.4. **Method used.** Along this paper, we use rational functions of the square of the lengths of the edges of the d-simplices and naturally some systems of polynomial equations appear. Computations with such systems requires Gröbner basis softwares. For this, we have used the J. C. Faugere's software "FGb" (see [7]) and the computer algebra system Maple (see [9]) which provides some other tools we needed.

2. Standard definitions and results about tetrahedra.

2.1. **General tetrahedra.** Let $\mathcal{T} = (ABCD)$ be a tetrahedron, G its centroid and $\mathcal{V}(\mathcal{T})$ its volume. We assume A, B, C, D are not coplanar and the circumsphere of $\mathcal{T}$ has center O and radius 1. Let $BC = a$, $CA = b$, $AB = c$, $AD = a'$, $BD = b'$, $CD = c'$.

Remark that the 3-tuple (a, b, c) does not play the same role as (a', b', c').

Proposition 1. *For all $\mathcal{T}$, we have*

i) $$OG^2 = 1 - \frac{1}{16}(a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2)$$
$$= 1 - \frac{1}{4}(GA^2 + GB^2 + GC^2 + GD^2).$$ *(See [2] p. 64 and ii)).*

ii) $$GA^2 = \frac{3}{16}(a'^2 + b^2 + c^2) - \frac{1}{16}(a^2 + b'^2 + c'^2),$$
$$GB^2 = \frac{3}{16}(b'^2 + c^2 + a^2) - \frac{1}{16}(b^2 + c'^2 + a'^2),$$
$$GC^2 = \frac{3}{16}(c'^2 + a^2 + b^2) - \frac{1}{16}(c^2 + a'^2 + b'^2),$$
$$GD^2 = \frac{3}{16}(a'^2 + b'^2 + c'^2) - \frac{1}{16}(a^2 + b^2 + c^2).$$ *(see [11] p. xv).*

If $u, v, w \in \mathbb{R}^3$ then $Gram(u, v, w)$ refers to the Gram matrix of these three vectors. If U is a square matrix then $det(U)$ refers to its determinant.

Proposition 2. *Let* $\Gamma(A, B, C, D) = det \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & a'^2 \\ 1 & c^2 & 0 & a^2 & b'^2 \\ 1 & b^2 & a^2 & 0 & c'^2 \\ 1 & a'^2 & b'^2 & c'^2 & 0 \end{pmatrix}$ *be the Cayley-Menger determinant, and let* $\Delta(A, B, C, D) = det \begin{pmatrix} 0 & c^2 & b^2 & a'^2 \\ c^2 & 0 & a^2 & b'^2 \\ b^2 & a^2 & 0 & c'^2 \\ a'^2 & b'^2 & c'^2 & 0 \end{pmatrix}$.

With these notations, we have

i) The volume as a function of a, b, c, a', b', c' (see [3] p. 168):
$288 \times \mathcal{V}(\mathcal{T})^2 = 8 \times det(Gram(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})) = \Gamma(A, B, C, D)$.
Therefore, A, B, C, D are not coplanar if and only if $\Gamma(A, B, C, D) > 0$.

ii) Let R be the circumradius of the tetrahedron (here $R = 1$).
  a) Crelle's formula (1821): $6R\,\mathcal{V}(\mathcal{T}) = \delta$ where δ denotes the area of the triangle with sides of lengths aa', bb', cc' (see [2] p. 250)
  b) $R^2 = -\dfrac{\Delta(A, B, C, D)}{2\Gamma(A, B, C, D)}$ *(see [3] p. 168).*

iii) There exists a relation between a, b, c, a', b', c': det $\begin{pmatrix} 1/2 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & a'^2 \\ 1 & c^2 & 0 & a^2 & b'^2 \\ 1 & b^2 & a^2 & 0 & c'^2 \\ 1 & a'^2 & b'^2 & c'^2 & 0 \end{pmatrix} = 0.$

Proof. iii) comes from ii)b) and $R = 1$: $\Gamma(A, B, C, D) + \frac{1}{2}\Delta(A, B, C, D) = 0$. □

Proposition 3. *We assume $\mathcal{T}$ is a non planar tetrahedron or a cyclic quadrilateral. The expression $Pt(\mathcal{T}) = -\Delta(A, B, C, D)$ has the following properties:*

i) $Pt(\mathcal{T}) = (bb' + cc' - aa')(cc' + aa' - bb')(aa' + bb' - cc')(aa' + bb' + cc')$
$= 2a^2a'^2b^2b'^2 + 2b^2b'^2c^2c'^2 + 2c^2c'^2a^2a'^2 - a^4a'^4 - b^4b'^4 - c^4c'^4.$

ii) If $\mathcal{T}$ is not planar then each factor of the product appearing in i) is positive (Ptolemy's inequality, see [5]*). (see also* [10] *p.549,555).*
Moreover $Pt(\mathcal{T}) = 576 \times \mathcal{V}(\mathcal{T})^2 \times R^2$ (From Proposition 2).

iii) If $\mathcal{T}$ is a cyclic quadrilateral then $Pt(\mathcal{T}) = 0$.

Conversely let a, b, c, a', b', c' be six positive reals. Does there exist a tetrahedron such that the lengths of its edges are these reals?

Proposition 4. *The necessary and sufficient condition that the six lengths form a tetrahedron (eventually planar) appears to be:*

i) a, b, c are the lengths of a triangle that is $16 \times S^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \geq 0$ (then S is the area of this triangle).

ii) det $\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & a'^2 \\ 1 & c^2 & 0 & a^2 & b'^2 \\ 1 & b^2 & a^2 & 0 & c'^2 \\ 1 & a'^2 & b'^2 & c'^2 & 0 \end{pmatrix} \geq 0$

Proof. (See also [10] p. 545) Assertion $i)$ of Proposition 3 implies that the condition $ii)$ is necessary.
The following can be generalized in any dimension:
The tetrahedron exists if and only if $Gram(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})$ is a symmetric non negative matrix. That gives the conditions $i), ii)$. □

2.2. **Isosceles (or equifacetal) tetrahedra.**

Definition. *Let $\mathcal{T}$ be a non planar tetrahedron whose circumradius is 1. The tetrahedron $\mathcal{T}$ is said to be isosceles (or equifacetal) if $a = a', b = b', c = c'$.*

Proposition 5. *We have*

i) $\mathcal{T}$ is isosceles if and only if $G = O$. (see [1] *p. 197).*

ii) If $\mathcal{T}$ is isosceles then $72\,\mathcal{V}^2(\mathcal{T}) = (b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)$ and $a^2 < b^2 + c^2$, $b^2 < c^2 + a^2$, $c^2 < a^2 + b^2$. (see [8] *p. 205).*

iii) For all $\mathcal{T}$, $0 < aa' + bb' + cc' \leq 8$. Moreover $aa' + bb' + cc' = 8$ if and only if $\mathcal{T}$ is isosceles. (see [10] *p. 558).*

The following result is a direct consequence of propositions 5 and 1.

Proposition 6. *Let $\mathcal{T} = (ABCD)$ be a tetrahedron eventually planar whose centroid is G. $GA = GB = GC = GD$ if and only if $\mathcal{T}$ is an isosceles tetrahedron or a planar rectangle.*

3. Deformation from $\mathcal{T}_0$ to $\mathcal{T}_1$ $(d = 3)$

3.1. **Parameters and notations.** We adapt our preceding notations; A_0, B_0, C_0, D_0 are four points of $\mathcal{S}$. We assume only A_0, B_0, C_0, D_0 are not all equal. Then the functions ϕ, ψ are continuous in $\mathcal{T}_0$.

Remark. *i) The following calculations are valid if* A_0, B_0, C_0, D_0 *are four points, not all equal, of* $\mathcal{C}$, *a planar circle of center* O *and radius* 1.
ii) Let $\Delta = \{(P,P,P,P) : P \in \mathcal{S}\}$. ϕ *is defined on* $\mathcal{S}^4 \backslash \Delta$.

Definition. *The parameters* $(d_{ij})_{i<j}$ *of* $\mathcal{T}_0$ *are the square of the lengths of their edges:* $d_{12} = A_0B_0{}^2, d_{13} = A_0C_0{}^2, d_{14} = A_0D_0{}^2, d_{23} = B_0C_0{}^2, d_{24} = B_0D_0{}^2, d_{34} = C_0D_0{}^2$. *Let us recall that they are linked by a polynomial equality (see Assertion iii) of Proposition 2) and two polynomial inequalities (see Proposition 4).*

Let $g_1 = G_0A_0{}^2, g_2 = G_0B_0{}^2, g_3 = G_0C_0{}^2, g_4 = G_0D_0{}^2$ and let $p_0 = 1 - OG_0{}^2 = \dfrac{1}{4}(g_1 + g_2 + g_3 + g_4)$ and p_1 be the opposites of the powers of G_0 and G_1 with respect to $\mathcal{S}$. According to Proposition 1, $(g_i)_i$ and p_0 are polynomial functions of $(d_{ij})_{ij}$.

Proposition 7. *We have*

i) The parameters of $\mathcal{T}_1$ *are* $\left(d^1_{ij} = p_0{}^2 \dfrac{d_{ij}}{g_ig_j}\right)_{i<j}$.

ii) $p_1 = \dfrac{p_0{}^2}{16} \sum_{i<j} \dfrac{d_{ij}}{g_ig_j}$.

iii) $OG_0{}^2$ *and* $OG_1{}^2$ *are rational functions of the* $(d_{ij})_{ij}$.

iv) $\mathcal{V}(\mathcal{T}_0)^2$ *and* $\mathcal{V}(\mathcal{T}_1)^2$ *are rational functions of the* $(d_{ij})_{ij}$.

Proof. For $i)$: $G_0A_0 \times G_0A_1 = p_0$ and $G_0A_1{}^2 = \dfrac{p_0{}^2}{g_1}$. The triangles $(G_0A_0B_0)$ and $(G_0B_1A_1)$ are similar; then $\dfrac{A_1B_1}{A_0B_0} = \dfrac{G_0A_1}{G_0B_0}$ and $A_1B_1{}^2 = d_{12}\dfrac{p_0{}^2}{g_1g_2}$.

We deduce $ii)$ from $i)$ and $iii)$ from $ii)$. We deduce $iv)$ from $i)$ and Proposition 2. $i)$. □

3.2. **Inequalities.**

Let $\mathcal{T}$ be a tetrahedron or a cyclic quadrilateral.

Let $\mathcal{P}(\mathcal{T})$ be the property: "the edges of $\mathcal{T}$ satisfy $d_{12} = d_{34}, d_{13} = d_{24}, d_{14} = d_{23}$" i.e. "$\mathcal{T}$ is an isosceles tetrahedron or a planar rectangle".

The first key is the following result:

Theorem 1. *We have the following inequalities:*

i) $OG_1 \le OG_0$;

ii) for any $(ijkl) \in \{(1234), (1324), (1423)\}$, $d_{ij}d_{kl} \le d^1_{ij}d^1_{kl}$;

iii) $Pt(\mathcal{T}_0) \le Pt(\mathcal{T}_1)$ *(see Proposition 3);*

Moreover, for i), ii) and iii), equalities stands if and only if $\mathcal{P}(\mathcal{T}_0)$ *holds.*

Proof. • For $i)$: the following choice of unknowns enables us to conclude; we may assume $g_1 - g_2 = s_1, g_2 - g_3 = s_2, g_3 - g_4 = s_3, g_4s_4 = 1$ where $s_1, s_2, s_3 \ge 0$ and $s_4 > 0$. Let's set $E = 64\dfrac{(p_1 - p_0)g_1g_2g_3}{p_0}$.

We note that $signum(E) = signum(OG_0{}^2 - OG_1{}^2)$ and we obtain, using the J. C. Faugere's software "FGb", this miraculous result:
$E = d_{23}(s_1 - s_3)^2 + 16s_2{}^2s_3{}^2s_4 + 4d_{34}s_1s_2 + 12s_1s_2{}^2 + 4s_1s_2s_3 + 20s_2s_3{}^2 + 21s_2s_3{}^3s_4 + d_{24}s_3{}^2 + d_{34}s_1{}^2 + 4s_1s_3{}^2 + d_{24}s_1{}^2s_3s_4 + d_{34}s_1{}^2s_3s_4 + 12s_3{}^3 + 2d_{24}s_1s_2s_3s_4 + 8s_2{}^3 + 3d_{24}s_1s_3{}^2s_4 + 6d_{34}s_1s_2s_3s_4 + 4d_{34}s_2{}^2 + d_{34}s_3{}^2 + 5d_{34}s_2{}^2s_3s_4 + d_{24}s_1{}^2 + 3d_{34}s_1s_2{}^2s_4 + 4s_2{}^3s_3s_4 + 3d_{34}s_2s_3{}^2s_4 + 2d_{24}s_1s_3 + 12s_2{}^2s_3 + 2d_{34}s_1s_3 + 4d_{34}s_2s_3 + 9s_3{}^4s_4 + 2d_{34}s_2{}^3s_4 + d_{34}s_1{}^2s_2s_4 + s_1{}^2s_2s_3s_4 + 4s_1s_2{}^2s_3s_4 + s_1{}^2s_3{}^2s_4 + 10s_1s_2s_3{}^2s_4 + 6s_1s_3{}^3s_4 + 3d_{34}s_1s_3{}^2s_4$; obviously E is a non negative real.

Moreover if $E = 0$ then $s_1 = s_2 = s_3 = 0$ and, according to Proposition 6, $\mathcal{P}(\mathcal{T}_0)$ holds.

• For $ii)$: let $\Lambda = \dfrac{p_0^4}{g_1g_2g_3g_4}$. By Proposition 7, $d^1_{ij}d^1_{kl} = \Lambda d_{ij}d_{kl}$. According to the AM-GM inequality, $\Lambda \ge 1$ and $d_{ij}d_{kl} \le d^1_{ij}d^1_{kl}$.

Moreover if $d_{ij}d_{kl} = d^1_{ij}d^1_{kl}$ then $\Lambda = 1$ and according to the properties of the AM-GM inequality, $g_1 = g_2 = g_3 = g_4$ and $\mathcal{P}(\mathcal{T}_0)$ holds.
• For iii): An easy computation gives $Pt(\mathcal{T}_1) = \Lambda^2 Pt(\mathcal{T}_0)$ and we reason as for ii). $\square$

4. Solution of the case $d = 3$. Part 1

The $(d^n_{ij})_{i<j}$ and $(d_{ij})_{i<j}$ refer to the parameters of $\mathcal{T}_n$ and $\mathcal{T}_0$.

Theorem 2. *i) The sequence $(OG_n)_n$ tends to 0.*
ii) Let $\mathcal{T} = (ABCD)$ be a cluster point of the bounded sequence $(\mathcal{T}_i)$. Then $\mathcal{T}$ is not flat and is isometric to a fixed isosceles tetrahedron that admits the following parameters:
$d^{\infty}_{12}{}^2 = Ld_{12}d_{34}, d^{\infty}_{13}{}^2 = Ld_{13}d_{24}, d^{\infty}_{14}{}^2 = Ld_{14}d_{23}$ *where*
$L = \dfrac{64}{(\sqrt{d_{12}d_{34}} + \sqrt{d_{13}d_{24}} + \sqrt{d_{14}d_{23}})^2}$.

Proof. According to Theorem 1 iii), the sequence $(Pt(\mathcal{T}_i))$ is increasing, then is positive and, for all i, A_i, B_i, C_i, D_i are not coplanar; for all i, $G_i \notin \{A_i, B_i, C_i, D_i\}$ then the sequence $(\mathcal{T}_i)$ is well defined and the functions ϕ, ψ are continuous in $(\mathcal{T}_i)$. Moreover the bounded sequence $(Pt(\mathcal{T}_i))$ converges to $Pt > 0$.

According to Theorem 1 i), the bounded sequence (OG_n) is decreasing and converges to $r \geq 0$. We can extract a subsequence $(\mathcal{T}_{n_k})_k$ such that $(A_{n_k})_k$, $(B_{n_k})_k$, $(C_{n_k})_k$, $(D_{n_k})_k$ converge to A, B, C, D. Let $\mathcal{T}$ be the tetrahedron $(ABCD)$ and G its centroid. Therefore $(G_{n_k})_k$ converges to G, $OG = r$ and $Pt(\mathcal{T}) = Pt$; thus A, B, C, D are not coplanar and $\phi(\mathcal{T}) = \mathcal{T}' = (A'B'C'D')$ is well defined; let G' be its centroid.

Assume $\mathcal{T}$ is not isosceles; then Theorem 1 i) implies that $OG' < OG$. Let $\epsilon \in (0, OG - OG')$ and let $\alpha > 0$ such that if $||\mathcal{T} - \mathcal{T}_n|| < \alpha$ then $OG_{n+1} - OG' < \epsilon$. There exists an integer n_k such that $||\mathcal{T} - \mathcal{T}_{n_k}|| < \alpha$; then $OG_{n_k+1} - OG' < \epsilon$. Thus $OG_{n_k+1} < OG = r$, a contradiction. Therefore $\mathcal{T}$ is isosceles and $(OG_n)_n$ converges to 0. A parameter d^{∞}_{ij} of $\mathcal{T}$ is the limit of the sequence $(d^{n_k}_{ij})_k$.

According to the proof of Theorem 1 ii), if $\{i, j, k, l\}$ is a permutation of $\{1, 2, 3, 4\}$, then the bounded sequence $(d^n_{ij}d^n_{kl})_n$ satisfy $d^{n+1}_{ij}d^{n+1}_{kl} = \Lambda_n d^n_{ij}d^n_{kl}$; it is increasing then is convergent. Thus the infinite product $\Pi^{\infty}_{n=0}\Lambda_n$ converges to $L \geq 1$ such that $d^{\infty}_{ij}{}^2 = Ld_{ij}d_{kl}$.
$d^{\infty}_{12} + d^{\infty}_{13} + d^{\infty}_{14} = 8$ gives the explicit value of L as a function in the (d_{ij}).
Thus there exists an unique value of d^{∞}_{ij} that is valid for all cluster points. $\square$

Corollary 1. *If $i < j$ then the sequence $(d^n_{ij})_n$ converges to d^{∞}_{ij}.*

Proof. Let $\mathcal{U} = \{(1234), (1324), (1423)\}$. If $(ijkl) \in \mathcal{U}$ then $d^n_{ij}d^n_{kl}$ converges to $d^{\infty}_{ij}{}^2$ and $\sum_{(ijkl)\in\mathcal{U}} \sqrt{d^n_{ij}d^n_{kl}}$ converges to 8.
$2\sum_{(ijkl)\in\mathcal{U}} \sqrt{d^n_{ij}d^n_{kl}} \leq \sum_{i<j} d^n_{ij} \leq 16$; thus $\sum_{(ijkl)\in\mathcal{U}} \left(\sqrt{d^n_{ij}} - \sqrt{d^n_{kl}}\right)^2$ converges to 0 and if $(ijkl) \in \mathcal{U}$ then $d^n_{ij} - d^n_{kl}$ converges to 0; therefore d^n_{ij} and d^n_{kl} converge to d^{∞}_{ij}. $\square$

It remains to prove that the $(\mathcal{T}_{2i})$ cannot turn around O.

5. Solution of the case $d = 3$. Part 2

We assume $\mathcal{T}_0$ is not isosceles. We study the convergence speed of the sequence $({OG_n}^2)$.

Definition. *Let f_n, g_n be two positive sequences.*
1. *We say that $f_n = \Theta(g_n)$ if and only if there exist two positive reals α, β such that, for all sufficiently large n, $\alpha g_n \leq f_n \leq \beta g_n$.*
2. *We say that $f_n \sim g_n$ if and only if $\lim_{n\to\infty} \dfrac{f_n}{g_n} = 1$.*

5.1. **Taylor series I.** Let $d_{ij}^n = d_{ij}^\infty + h_{ij}^n$, $h_n = (h_{ij}^n)_{i<j}$ (the sequence (h_n) converges to the zero vector) and $\delta_n = \sum_{i<j} h_{ij}^n$. Let $\epsilon_n = (h_{12}^n - h_{34}^n)^2 d_{12}^\infty + (h_{13}^n - h_{24}^n)^2 d_{13}^\infty + (h_{14}^n - h_{23}^n)^2 d_{14}^\infty$.

Proposition 8. $OG_{n+1}{}^2 = \dfrac{-\delta_n}{16} - \dfrac{1}{16^2}\epsilon_n + O(||h_n||^3)$.

Proof. Let us recall that $OG_n{}^2 = \dfrac{-\delta_n}{16}$. With the help of the software "FGb" we put $OG_{n+1}{}^2$ in the form of a rational function in the unknowns (h_{ij}^n) and we deduce the terms of degree at most two of its Taylor series:

$$OG_{n+1}{}^2 = \frac{-(\delta_n + \tau_n)}{16\left(1 + \dfrac{\delta_n}{4}\right)} + O\left(||h_n||^3\right) = \frac{-\delta_n}{16} - \frac{1}{16}\left(\tau_n - \frac{\delta_n{}^2}{4}\right) + O(||h_n||^3) \text{ where:}$$

$$\tau_n - \frac{\delta_n{}^2}{4} = \frac{1}{16}\left((h_{12}^n - h_{34}^n)^2 d_{12}^\infty + (h_{13}^n - h_{24}^n)^2 d_{13}^\infty + (h_{14}^n - h_{23}^n)^2 d_{14}^\infty\right) + O(||h_n||^3).$$ □

5.2. **Taylor series II.** Let us recall that $d_{14}^\infty = 8 - d_{12}^\infty - d_{13}^\infty$ and $d_{12}^\infty + d_{13}^\infty > 4$.

Now we prove the second key.

Proposition 9. *i)* $\delta_n = O(||h_n||^2)$.
ii) There exists $k > 0$ *such that* $\epsilon_n \geq -k\delta_n$.

Proof. For i): From assertion *iii)* of proposition 2, h_n satisfies an algebraic relation. We obtain with Maple the Taylor series of the preceding relation; we consider the terms of degree at most two: $4(4 - d_{12}^\infty)(4 - d_{13}^\infty)(d_{12}^\infty + d_{13}^\infty - 4)\delta_n + \sigma_n' + O(||h_n||^3) = 0$ where $\sigma_n' = O(||h_n||^2)$ is a non negative quadratic form in the (h_{ij}). Thus $\delta_n = O(||h_n||^2)$.

For ii): If we take $h_{12}^n = -h_{13}^n - h_{14}^n - h_{23}^n - h_{24}^n - h_{34}^n$, then we transform the expression σ_n' in σ_n that satisfies : $\sigma_n = -4(4 - d_{12}^\infty)(4 - d_{13}^\infty)(d_{12}^\infty + d_{13}^\infty - 4)\delta_n + O(||h_n||^3)$. σ_n is a quadratic form in $h_n' = (h_{13}^n, h_{14}^n, h_{23}^n, h_{24}^n, h_{34}^n) \in \mathbb{R}^5$ whose symmetric associated matrix Σ is defined by:

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{12} & \Sigma_{14} & 2\Sigma_{12} \\ * & \Sigma_{22} & 2\Sigma_{12} - \Sigma_{22} & \Sigma_{12} & 2\Sigma_{12} \\ * & * & \Sigma_{22} & \Sigma_{12} & 2\Sigma_{12} \\ * & * & * & \Sigma_{11} & 2\Sigma_{12} \\ * & * & * & * & 4\Sigma_{12} \end{pmatrix}$$

$$\text{with} \begin{cases} \Sigma_{11} = 2d_{12}^\infty d_{13}^\infty + 32 - 12d_{13}^\infty + {d_{13}^\infty}^2 - 12d_{12}^\infty + {d_{12}^\infty}^2, \\ \Sigma_{12} = -4d_{12}^\infty + 16 + d_{12}^\infty d_{13}^\infty - 8d_{13}^\infty + {d_{13}^\infty}^2, \\ \Sigma_{14} = -{d_{12}^\infty}^2 + 4d_{12}^\infty + {d_{13}^\infty}^2 - 4d_{13}^\infty, \\ \Sigma_{22} = -4d_{13}^\infty + {d_{13}^\infty}^2. \end{cases}$$

By the same way we transform the expression ϵ_n in ϵ_n' such that $\epsilon_n = \epsilon_n' + O(||h_n||^3)$. ϵ_n' is a quadratic form in h_n' whose matrix is:

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{12} & E_{14} & 2E_{12} \\ * & E_{22} & 2E_{12} - E_{22} & E_{12} & 2E_{12} \\ * & * & E_{22} & E_{12} & 2E_{12} \\ * & * & * & E_{11} & 2E_{12} \\ * & * & * & * & 4E_{12} \end{pmatrix}$$

$$\text{with} \begin{cases} E_{11} = d_{12}^\infty + d_{13}^\infty \\ E_{12} = d_{12}^\infty \\ E_{14} = d_{12}^\infty - d_{13}^\infty \\ E_{22} = 8 - d_{13}^\infty \end{cases}$$

Let F be the plane of $\mathbb{R}^5$ defined by the relations: $\{h_{13}^n = h_{24}^n, h_{14}^n = h_{23}^n, h_{34}^n = -h_{13}^n - h_{14}^n\}$. F is the nullspace of the matrices Σ and E.

Definition. *We say that* h_n *has an acceptable value if and only if the* $(d_{ij}^n) = (d_{ij}^\infty + h_{ij}^n)$ *satisfy Proposition 2 iii).*

Assume h_n has an acceptable value and a small norm; an easy computation proves that $h_n' \in F$ if and only if $\mathcal{T}_n$ is an isosceles tetrahedron.

We know that if $\mathcal{T}_0$ is not isosceles then, for all n, the tetrahedron $\mathcal{T}_n$ is not isosceles. Thus here

$h'_n \notin F$ and if $h'_n = u_n + v_n$ is the decomposition associated to $\mathbb{R}^5 = F \oplus F^\perp$, where $F^\perp$ is the orthogonal of F, then (v_n) tends to 0 and, for all n, $v_n \neq 0$. Moreover ${h'_n}^T \Sigma h'_n = {v_n}^T \Sigma v_n > 0$ and ${h'_n}^T E h'_n = {v_n}^T E v_n > 0$.

$spectrum(E_{|F^\perp}) = \{2d_{13}^\infty, 8d_{12}^\infty, 2d_{14}^\infty\}$ and the associated eigenvectors are $[-1,0,0,1,0]^T$, $[1,1,1,1,2]^T$ and $[0,-1,1,0,0]^T$.
$spectrum(\Sigma_{|F^\perp}) = \{2(4-d_{12}^\infty)(4-d_{14}^\infty), 8(4-d_{13}^\infty)(4-d_{14}^\infty), 2(4-d_{12}^\infty)(4-d_{13}^\infty)\}$ and the associated eigenvectors are also $[-1,0,0,1,0]^T, [1,1,1,1,2]^T, [0,-1,1,0,0]^T$ (E and Σ are simultaneously diagonalizable).
Let $m = \min\left\{\frac{d_{13}^\infty}{(4-d_{12}^\infty)(4-d_{14}^\infty)}, \frac{d_{12}^\infty}{(4-d_{13}^\infty)(4-d_{14}^\infty)}, \frac{d_{14}^\infty}{(4-d_{12}^\infty)(4-d_{13}^\infty)}\right\}$,
$M = \max\left\{\frac{d_{13}^\infty}{(4-d_{12}^\infty)(4-d_{14}^\infty)}, \frac{d_{12}^\infty}{(4-d_{13}^\infty)(4-d_{14}^\infty)}, \frac{d_{14}^\infty}{(4-d_{12}^\infty)(4-d_{13}^\infty)}\right\}$.

Thus if h_n has an acceptable value then $m \leq \frac{\epsilon'_n(h'_n)}{\sigma_n(h'_n)} \leq M$ and

$$\epsilon'_n = \epsilon_n + O(||h_n||^3) \geq m\sigma_n = -4m(4-d_{12}^\infty)(4-d_{13}^\infty)(4-d_{14}^\infty)\delta_n + O(||h_n||^3).$$

To finish the proof of ii), it remains to show that $\{\epsilon_n$ and δ_n are $\Theta(||h_n||^2)\}$ $(*)$.
Indeed $(*)$ implies that for all sufficiently large n, $\epsilon_n > -k\delta_n$, choosing a positive number $k < 4\rho$ with $\rho = \min\{d_{13}^\infty(4-d_{13}^\infty), d_{12}^\infty(4-d_{12}^\infty), d_{14}^\infty(4-d_{14}^\infty)\}$. □

Remark. *i) If the limit form is a regular tetrahedron then* $m = M = \frac{3}{2}$.
ii) It can be proved that $\rho \leq \frac{32}{9}$ *with equality if and only if the limit form is regular.*

In the following we prove the property $(*)$.

5.3. **Improvement of certain estimates.**

Lemma 1. *If* $(ijkl) \in \mathcal{U} = \{(1234), (1324), (1423)\}$ *then* $d_{ij}^n d_{kl}^n - (d_{ij}^\infty)^2 = O(||h_n||^2)$.

Proof. Let $L_n = \frac{64}{(\sqrt{d_{12}^n d_{34}^n} + \sqrt{d_{13}^n d_{24}^n} + \sqrt{d_{14}^n d_{23}^n})^2}$. (L_n) tends to 1.
We have $(d_{ij}^\infty)^2 = L_n d_{ij}^n d_{kl}^n$, $d_{ij}^n d_{kl}^n - (d_{ij}^\infty)^2 = (\frac{1}{L_n} - 1)(d_{ij}^\infty)^2$ and, consequently, $\frac{1}{L_n} - 1 = \frac{(\sqrt{d_{12}^n d_{34}^n} + \sqrt{d_{13}^n d_{24}^n} + \sqrt{d_{14}^n d_{23}^n})^2 - 8^2}{64}$. Thus, if we set $u_n = \sqrt{d_{12}^n d_{34}^n} + \sqrt{d_{13}^n d_{24}^n} + \sqrt{d_{14}^n d_{23}^n} - 8$, we have to prove that $u_n = O(||h_n||^2)$:

$$\begin{aligned}
u_n &= \textstyle\sum_{(ijkl)\in\mathcal{U}} \sqrt{(d_{ij}^\infty)^2 + d_{ij}^\infty(h_{ij}^n + h_{kl}^n) + O(||h_n||^2)} - 8 \\
&= \textstyle\sum_{(ijkl)\in\mathcal{U}} d_{ij}^\infty \sqrt{1 + \frac{h_{ij}^n + h_{kl}^n}{d_{ij}^\infty} + O(||h_n||^2)} - 8 \\
&= \textstyle\sum_{(ijkl)\in\mathcal{U}} d_{ij}^\infty + \frac{h_{ij}^n + h_{kl}^n}{2} + O(||h_n||^2) - 8 \\
&= \frac{\sum_{i<j} h_{ij}}{2} + O(||h_n||^2) \\
&= = O(||h_n||^2), \text{ by proposition 9.}
\end{aligned}$$

□

Lemma 2. *If* $(ijkl) \in \mathcal{U}$ *then* $h_{ij}^n + h_{kl}^n = O(||h_n||^2)$.

Proof. $d_{ij}^n d_{kl}^n - (d_{ij}^\infty)^2 = d_{ij}^\infty(h_{ij} + h_{kl}) + h_{ij}h_{kl} = O(||h_n||^2)$ by Lemma 1. □

Lemma 3. *For all sufficiently large* n *there exists* $(ijkl) \in \mathcal{U}$ *such that*
$|h_{ij}^n - h_{kl}^n| \geq \frac{1}{\sqrt{6}}||h_n||$.

Proof. By Lemma 2 there exists $A>0$ such that, for all n and $(ijkl)\in\mathcal{U}$, $|h^n_{ij}+h^n_{kl}|\leq A||h_n||^2$. Let $\epsilon\in(0,\dfrac{1}{6A})$. There exists N such that $n\geq N\Rightarrow ||h_n||<\epsilon$. Let $n\geq N$ be a fixed integer. We may assume $|h^n_{12}|=\sup_{i<j}|h^n_{ij}|$. Thus $|h^n_{12}|\geq\dfrac{||h_n||}{\sqrt{6}}\geq\dfrac{|h^n_{12}|}{\sqrt{6}}$ and $|h^n_{12}+h^n_{34}|\leq 6A(h^n_{12})^2$ and $|h^n_{12}-h^n_{34}|\geq 2|h^n_{12}|-|h^n_{12}+h^n_{34}|\geq 2|h^n_{12}|-6A(h^n_{12})^2$.
$0<|h^n_{12}|\leq||h_n||<\epsilon<\dfrac{1}{6A}$ then $6A(h^n_{12})^2<|h^n_{12}|$ and $|h^n_{12}-h^n_{34}|\geq|h^n_{12}|\geq\dfrac{1}{\sqrt{6}}||h_n||$. □

Proposition 10. *If (h_n) has acceptable values, then for all sufficiently large n, $\epsilon_n\geq\lambda||h_n||^2$ with $\lambda=\dfrac{\inf_{i<j}d^{\infty}_{ij}}{6}$; moreover $-\delta_n=\Theta(||h_n||^2)$.*

Proof. $\epsilon_n=(h^n_{12}-h^n_{34})^2d^{\infty}_{12}+(h^n_{13}-h^n_{24})^2d^{\infty}_{13}+(h^n_{14}-h^n_{23})^2d^{\infty}_{14}$ and Lemma 3 give the first part. For the second part, if h_n has an acceptable value then we know, by proposition 9, that:
i) $\epsilon'_n=\epsilon_n+O(||h_n||^3)$. Thus, by the first part, if $\mu<\lambda$ then for all sufficiently large n, $\epsilon'_n\geq\mu||h_n||^2$.
ii) For all n, $\epsilon'_n(h'_n)\leq M\sigma_n(h'_n)$. Then $\sigma_n\geq\dfrac{\mu}{M}||h_n||^2$ (we can see σ_n as a function of h_n) for all sufficiently large n.
iii) $-\delta_n=\dfrac{\sigma_n}{\nu}+O(||h_n||^3)$ with $\nu=4(4-d^{\infty}_{12})(4-d^{\infty}_{13})(4-d^{\infty}_{14})$.
Thus if $\mu_1<\lambda$, then for all sufficiently large n, $-\delta_n\geq\dfrac{\mu_1}{M\nu}||h_n||^2$. □

5.4. **The main result in dimension three.**
Let $r=\max\left\{\dfrac{|d^{\infty}_{12}-2|}{2},\dfrac{|d^{\infty}_{13}-2|}{2},\dfrac{|d^{\infty}_{14}-2|}{2}\right\}\in\left[\frac{1}{3},1\right)$.

Theorem 3. *The sequences $(\mathcal{T}_{2i})_{i\in\mathbb{N}}$ and $(\mathcal{T}_{2i+1})_{i\in\mathbb{N}}$ are well defined and converge to two non planar isosceles tetrahedra that are symmetric with respect to O. Moreover the convergence is with at least geometric speed.*

Proof. ${OG_{n+1}}^2=\dfrac{-\delta_n}{16}-\dfrac{1}{16^2}\epsilon_n+O(||h_n||^3)<\dfrac{-\delta_n}{16}+\dfrac{1}{16^2}k\delta_n+O(||h_n||^3)\sim {OG_n}^2\left(1-\dfrac{k}{16}\right)$ (according to the preceding proposition). Finally, for all sufficiently large n, $OG_{n+1}\leq qOG_n$ where $q<\sqrt{1-\dfrac{\rho}{4}}=r$.

For all sufficiently large n, G_n and G_{n+1} are close to O then close to the middle points of the segments $[A_nA_{n+1}]$ and $[A_{n+1}A_{n+2}]$; thus for all sufficiently large n, $A_nA_{n+2}\leq 3G_nG_{n+1}$. Let $p\in\mathbb{N}^*$; $A_nA_{n+2p}\leq 3\sum_{k=n}^{n+2p-2}G_kG_{k+1}\leq 3\sum_{k=n}^{n+2p-2}(OG_k+OG_{k+1})\leq 6\sum_{k=n}^{n+2p}OG_k$. The series $\sum OG_n$ converges, then $(A_{2n})_n$ is a Cauchy sequence; thus it converges to A^{∞}. By the same way $(A_{2n+1})_n$ converges to A'^{∞}, the symmetric of A^{∞} with respect to O. Moreover $A_{2n}A^{\infty}\leq 6\sum_{k=2n}^{\infty}OG_k\leq\dfrac{6}{1-q}OG_{2n}$ for all sufficiently large n; therefore $A_{2n}A^{\infty}=O(q^{2n})$ and the sequences $(\mathcal{T}_{2i})$ and $(\mathcal{T}_{2i+1})$ converge with at least geometric speed. □

5.5. **About the limit form.** We consider the system of barycentric coordinates, for $\mathbb{R}^3$, determined by the vertices A_0,B_0,C_0,D_0 of $\mathcal{T}_0$. Let(u,v,w,h) be four non all equal reals. $\{\alpha A_0+\beta B_0+\gamma C_0+\delta D_0\ ;\ u\alpha+v\beta+w\gamma+h\delta=0\ ,\ \alpha+\beta+\gamma+\delta=1\}$ is a plane Π.
(u,v,w,h) are said to be the barycentric coordinates of Π.

We use the notations of 2.1. $\mathcal{T}_0$ is not isosceles; then $\{a'bc,b'ca,c'ab,a'b'c'\}$ are valid coordinates and define a plane Π (Lemoine's plane) that does not intersect $\mathcal{S}$. Let I_1,I_2 be the two points satisfying the following conditions: they are symmetric with respect to Π and inverse with respect to $\mathcal{S}$. They are said to be the isodynamic points of $\mathcal{T}_0$. There exist only two inversions leaving invariant $\mathcal{S}$ and mapping A_0,B_0,C_0,D_0 on the vertices of an isosceles tetrahedron; I_1,I_2 are the centers of these inversions (see [11] p. 184-186). The link with our paper is that the associated isosceles tetrahedra (that are not symmetric with respect to O) are isometric to the limits of the sequence $(\mathcal{T}_i)$ but these couples are distinct.

Remark. • *The limit form is a regular tetrahedron if and only if the parameters of the non isosceles tetrahedron $\mathcal{T}_0$ satisfy $d_{12}d_{34} = d_{13}d_{24} = d_{14}d_{23} < \frac{64}{9}$ that is $\mathcal{T}_0$ is an isodynamic tetrahedron.*
• *According to numerical experiments, we conjecture that $OG_{n+1} \sim r \times OG_n$ (convergence with order one); in particular it can be observed the following facts:*

i) If the limit isosceles tetrahedron is almost flat, that is close to a rectangle, then the factor r is close to 1.

ii) If the limit form is a regular tetrahedron then $OG_{n+1} \sim \dfrac{1}{3} \times OG_n$.

6. Sequence of cyclic quadrilaterals

6.1. **Degenerate simplices.** Now we work in the Euclidean plane and we have a look on the case where $\mathcal{T}_0$ is a cyclic quadrilateral. We see the following as a degenerate case of the preceding study.

Theorem 4. *We consider a convex cyclic quadrilateral $\mathcal{T}_0 = (A_0B_0C_0D_0)$ with circumcircle $\mathcal{C}$, the circle of center O and radius 1, and such that its vertices are pairwise distinct; we use the preceding iteration and produce a sequence of convex quadrilaterals $(\mathcal{T}_i)_i$. The sequences $(\mathcal{T}_{2i})_i$ and $(\mathcal{T}_{2i+1})_i$ are well defined and converge to rectangles that have same image, whose centroid is O and whose lengths of the edges are $d_{13}^\infty = 4$, $d_{12}^\infty = 4\dfrac{\sqrt{d_{12}d_{34}}}{\sqrt{d_{13}d_{24}}}$ and $d_{14}^\infty = 4 - d_{12}^\infty$. If the limit form is not a square then $OG_{n+1} \sim \dfrac{|d_{12}^\infty - 2|}{2} OG_n$ and the sequence converges with at least geometric speed; else the convergence is even faster. We study examples where the limit form is a square and the convergence is with order three.*

We keep the preceding notations. In particular the parameters $(d_{ij}^n)_{i<j}$ refer to the square of the lengths of the edges (diagonals included) of $\mathcal{T}_n$.

Proof. • The beginning of the proof is similar to that of Theorem 2. We use Theorem 1 (note that inequality *iii*) is useless because $Pt(\mathcal{T}_n) = 0$); with *ii*) we prove that the sequences $(d_{ij}^n d_{kl}^n)_n$ are increasing and if $i < j$ then the $(d_{ij}^n)_n$ have a positive lower bound; thus a cluster point has pairwise distinct vertices. With *i*) we prove that a cluster point $\mathcal{T}$ is a rectangle. Assertion *ii*) of Theorem 2 is always valid and gives explicitly the $(d_{ij}^\infty)_{i<j}$, the parameters of $\mathcal{T}$.

The quadrilaterals are convex, then $d_{13}^\infty = 4$ and $d_{14}^\infty = 4 - d_{12}^\infty$. Let $\mu = d_{12}^\infty$.
• *Relations between the h_{ij}.*
Part I. There exist three algebraically independent relations between the 6 parameters $(d_{ij}^n)_{i<j}$. The computation of the terms of degree 1 of the Taylor series of the relation $\Gamma(A_n, B_n, C_n, D_n) = 0$ gives $h_{12}^n + h_{14}^n + h_{23}^n + h_{34}^n - h_{24}^n - h_{13}^n = O(||h_n||^2)$.

Moreover the triangles $(A_nB_nC_n)$, (B_n, C_n, D_n) admit a circumscribed circle of radius 1; for example for the first triangle the relation is ${a_n}^2{b_n}^2{c_n}^2 = 16 \times {S_n}^2$ (the value of the area S_n of the triangle $(A_nB_nC_n)$ is given in Proposition 4 i)). The computation of the terms of degree 1 of the Taylor series of these relations gives $h_{13}^n = O(||h_n||^2)$ and $h_{24}^n = O(||h_n||^2)$), thus $\delta_n = O(||h_n||^2)$.

Definition. *We say that h_n has an acceptable value if and only if the $(d_{ij}^n) = (d_{ij}^\infty + h_{ij}^n)$ satisfy the preceding three relations.*

Using the proof of Lemma 2, we show that $h_{12}^n + h_{34}^n = O(||h_n||^2)$ and $h_{14}^n + h_{23}^n = O(||h_n||^2)$. Thus ${h_{12}^n}^2 + {h_{14}^n}^2 = \Theta(||h_n||^2)$.
Part II. The computation of the terms of degree at most two of the Taylor series of the relations associated to the triangles $(A_nB_nC_n)$, $(B_nC_nD_n)$ give: $h_{13}^n\mu(4-\mu) + (h_{12}^n + h_{23}^n)^2 = O(||h_n||^3)$ and $h_{24}^n\mu(4-\mu) + (h_{23}^n + h_{34}^n)^2 = O(||h_n||^3)$ that is $h_{13}^n = \dfrac{-(h_{12}^n - h_{14}^n)^2}{\mu(4-\mu)} + O(||h_n||^3)$ and $h_{24}^n = \dfrac{-(h_{12}^n + h_{14}^n)^2}{\mu(4-\mu)} + O(||h_n||^3)$.

The relation $\Gamma(A_n,B_n,C_n,D_n)=0$ gives by a similar way
$\mu(\mu-4)(h^n_{12}+h^n_{14}+h^n_{23}+h^n_{34}-h^n_{24}-h^n_{13})+\mu {h^n_{14}}^2+(4-\mu){h^n_{12}}^2=O(||h_n||^3)$.
Using the last three relations we can deduce: $\delta_n=\dfrac{-{h^n_{14}}^2}{\mu}-\dfrac{{h_{12}}^2}{4-\mu}+O(||h_n||^3)$ and
$\delta_n=\Theta(||h_n||^2)$. Moreover $\epsilon_n=4\mu {h^n_{12}}^2+4(4-\mu){h^n_{14}}^2+O(||h_n||^3)$.
Finally ${OG_{n+1}}^2=\dfrac{(\mu-2)^2}{64}\left(\dfrac{{h^n_{14}}^2}{\mu}+\dfrac{{h_{12}}^2}{4-\mu}\right)+O(||h_n||^3)=\dfrac{(\mu-2)^2}{4}{OG_n}^2+O(||h_n||^3)$.
Case 1: $\mu\neq 2$; then the limit form is not a square and $OG_{n+1}\sim\dfrac{|\mu-2|}{2}OG_n$. The convergence is with order one.
Case 2: $\mu=2$; then the limit form is a square and $OG_{n+1}=O(||h_n||^{\frac{3}{2}})$ or $OG_{n+1}=O({OG_n}^{\frac{3}{2}})$. In fact the convergence is much faster than in the preceding case. For example if the polar angles of A_0,B_0,C_0,D_0 are $\{0,\arccos(0.923827833284),\arccos(-0.8),-\arccos(0.9)\}$ then we obtain a quasi square after few iterations.

Now we can conclude: in both cases the series $\sum_n OG_n$ converges; thus we may reason as in Theorem 3 and we obtain that the sequences $(\mathcal{T}_{2i})_{i\in\mathbb{N}}$ and $(\mathcal{T}_{2i+1})_{i\in\mathbb{N}}$ converge to rectangles that have same image with at least geometric speed. □

6.2. **A particular case.** The nullspace N of the symmetric matrix Σ is the hyperplane defined by $h^n_{13}=h^n_{24}$. Assume h_n has an acceptable value and a small norm; then a geometric argument or an algebraic computation shows that $h'_n\in N$ is equivalent to $\mathcal{T}_n$ is an isosceles trapezoid.

Proposition 11. *Preserving the assumptions of theorem 4, we assume there exists k such that $\mathcal{T}_k$ is an isosceles trapezoid but not a rectangle. If moreover $d^\infty_{12}=2$ then $OG_{n+1}\sim {OG_n}^3$.*

Proof. For all $n\geq k$, $\mathcal{T}_n$ is an isosceles trapezoid that admits OG_k as a line of symmetry. Therefore $\mathcal{T}_n$ cannot turn around O; we know explicitly the limit form of $\mathcal{T}_n$, then $\mathcal{T}_n$ converges to a rectangle that admits OG_k as a line of symmetry.

We study the rate of convergence using an explicit calculation. We may assume that the line of symmetry is the axis of abscissas. The vertical edges A_nD_n and B_nC_n have a_n,b_n as abscissas; then the abscissa of the centroid of $\mathcal{T}_n$ is $g_n=\dfrac{a_n+b_n}{2}$. An easy calculation gives:

$$a_{n+1}=-\frac{{a_n}^2b_n+2a_n{b_n}^2+{b_n}^3-4a_n}{{a_n}^2-2a_nb_n-3{b_n}^2+4},$$
$$b_{n+1}=-\frac{{b_n}^2a_n+2b_n{a_n}^2+{a_n}^3-4b_n}{{b_n}^2-2b_na_n-3{a_n}^2+4}.$$

We know that a_n+b_n tends to 0 and ${a_n}^2$ tends to the known expression $\dfrac{d^\infty_{12}}{4}$.

$$g_{n+1}\sim\frac{(a_n+b_n)(a_n^4-4{a_n}^3b_n-10{a_n}^2{b_n}^2+16{a_n}^2-4a_n{b_n}^3+16{b_n}^2-16+{b_n}^4)}{-32}.$$

i) $d^\infty_{12}\neq 2$. The limit form is not a square and we obtain $g_{n+1}\sim\left(1-\dfrac{d^\infty_{12}}{2}\right)g_n$ (convergence with order one). If $d^\infty_{12}>2$ then, for all sufficiently large n, $O\in]G_nG_{n+1}[$.

ii) $d^\infty_{12}=2$. The limit form is a square that is the parameters of $\mathcal{T}_0$ satisfy the relation $(d_{12})^2=d_{14}d_{23}$. Let $a_n=\dfrac{1}{\sqrt{2}}+u_n, b_n=-\dfrac{1}{\sqrt{2}}+v_n$ where u_n and v_n tend to 0 and $(u_n,v_n)\neq(0,0)$. We calculate the Taylor series of the preceding relation and we consider the terms of degree at most 2; we obtain $16\sqrt{2}(u_n-v_n)=-8(3u_n^2+3v_n^2+2u_nv_n)+O(||(u_n,v_n)||^3)$. Therefore $u_n-v_n\sim\dfrac{-1}{2\sqrt{2}}(3u_n^2+3v_n^2+2u_nv_n)$. $u_n-v_n=O(||(u_n,v_n)||^2)$ thus $u_n+v_n=\Theta(||(u_n,v_n)||)$.
We deduce easily the estimate $g_{n+1}\sim\dfrac{(a+b)(-4(u_n+v_n)^2+O(||(u_n,v_n)||^3)}{-32}\sim\dfrac{-4(u_n+v_n)^3}{-32}$ or $g_{n+1}\sim {g_n}^3$ (convergence with order three) what is astonishing.

For example if $a_0=0.955, b_0=0.12237784429$, then we obtain a quasi square after three iterations. □

6.3. **About the limit form.** As in the case of a tetrahedron there exist two inversions leaving $\mathcal{C}$ invariant and mapping A_0, B_0, C_0, D_0 on the vertices of a rectangle; the centers of these inversions are inverse with respect to $\mathcal{C}$. The associated rectangles are isometric to the limit of the sequence $(\mathcal{T}_i)$. In particular the limit form is a square if and only if the parameters of the non rectangular quadrilateral $\mathcal{T}_0$ satisfy $d_{12}d_{34} = d_{14}d_{23} < 4$ that is $\mathcal{T}_0$ is a harmonic (or isodynamic) quadrilateral.

Remark. *i) If the limit form is a square then the convergence seems to be with order three; more precisely we conjecture that* $OG_{n+1} \sim {OG_n}^3$.
ii) It can be observed a strange phenomenon: if $d = 3$ *and the limit form is close to a flat tetrahedron then we have seen that* $r \approx 1$. *If* $d = 2$ *then* r *is in general far from* 1*; in particular if the limit form is close to a square then* r *is close to* 0.

An explanation is that, if $d = 3$*, then* G_n *tends to* O *in such a way that the angles between* $\overrightarrow{OG_n}$ *and the edges of* $\mathcal{T}_n$ *are close to* $\frac{\pi}{2}$*; thus if* $d = 2$*, then we study a quasi orthogonal projection of the preceding* G_n *that converges faster than this* G_n.

7. Solution of the case $d = 2$

Now we consider a dynamical system of triangles.

7.1. **Some remarks.**

i) This problem has been solved in [4] by one of the authors. The proof in [4] is partially geometric; now we give a complete proof that is essentially algebraic. Moreover we give in Proposition 13 a much better estimate of OG_n.
ii) ϕ is not one to one: indeed, if $\mathcal{T}_1$ is a generic triangle, then there exists two triangles $\mathcal{T}_0$ such that $\phi(\mathcal{T}_0) = \mathcal{T}_1$.

7.2. **The parameters.** Let $a = B_0C_0, b = B_0C_0, c = C_0A_0$; the parameters of $\mathcal{T}_0$ are: $s = a^2 + b^2 + c^2, t = a^2b^2 + b^2c^2 + c^2a^2, u = a^2b^2c^2$.

Let us recall that the circumradius of $\mathcal{T}_0$ is 1; then $u = 4t - s^2$ and $t > \frac{s^2}{4}$. Moreover $u = 16 \times S^2$ where S is the area of $(A_0B_0C_0)$.
${OG_0}^2 = 1 - \frac{s}{9}$ and $0 < s \leq 9$. Therefore $s = 9 \Leftrightarrow G_0 = O \Leftrightarrow (A_0B_0C_0)$ is an equilateral triangle.
$s^2 - 3t = a^4 - a^2(b^2 + c^2) + b^4 + c^4 - b^2c^2$; this is a polynomial in a^2 with discriminant $-3(b^2 - c^2)^2 \leq 0$. Thus $t \leq \frac{s^2}{3}$ and $t = \frac{s^2}{3} \Leftrightarrow (A_0B_0C_0)$ is equilateral.

7.3. **Inequalities.** Here A_0, B_0, C_0 can be on a line but are not all equal; thus $s > 0, t > 0$. s_1, t_1, u_1 are the parameters of $\mathcal{T}_1$; we obtain:
$s_1 = \dfrac{s^2(6t - s^2)}{D}, t_1 = \dfrac{s^4t(9t - 2s^2)}{D^2}, u_1 = 4t_1 - {s_1}^2$ where
$D = -4s^3 + 18st - 108t + 27s^2$. $D > 0$ because the numerator of s_1 is positive; moreover D is bounded.

- $s_1 - s = \dfrac{3s(9 - s)(4t - s^2)}{D} \geq 0$. Moreover $s_1 = s$ if and only if $s = 9$ or $u = 0$ that is $A_0B_0C_0$ is equilateral or flat.
- $t_1 - t = \dfrac{9t(4t - s^2)(9(\frac{s^2}{3} - t)(s - 6)^2 + s^2(9 - s)(s - 3))}{D^2}$. If $s \geq 3$ then $t_1 \geq t$.
- $\dfrac{u_1}{u} = \left(\dfrac{s^3}{D}\right)^2$. $u_1 \geq u \Leftrightarrow s^3 \geq D \Leftrightarrow \nu = s^2(9 - s) + 18\left(\dfrac{s^2}{3} - t\right)(s - 6) \geq 0$.

If $s \geq 6$ then $\nu \geq 0$. Now we assume $s < 6$; $t > \dfrac{s^2}{4}$ implies that $\nu \geq \dfrac{s^3}{2}$.
Therefore $u_1 \geq u$.

7.4. **Convergence of the triangles.** We assume $\mathcal{T}_0$ is not an equilateral triangle. s_n, t_n, u_n refer to the parameters of $\mathcal{T}_n$.

Proposition 12. *Let $\mathcal{T}$ be a cluster point of the bounded sequence $(\mathcal{T}_n)_n$; then $\mathcal{T}$ is a non flat equilateral triangle. Moreover the lengths of the edges of $\mathcal{T}_n$ converge to $\sqrt{3}$.*

Proof. • The sequence (u_n) is increasing; thus it converges to $u^\infty > 0$. Let $\mathcal{T}$ be a cluster point of the sequence $(\mathcal{T}_n)$. The sequence (s_n) is increasing; thus it converges to $s^\infty > 0$. With a proof similar to that used in the theorem 3 we show that:

(1) s^∞, u^∞ are parameters of $\mathcal{T}$; therefore $\mathcal{T}$ is not flat.
(2) $\mathcal{T}$ is an equilateral triangle and $s^\infty = 9, u^\infty = 27$.

• Moreover for all sufficiently large n, $s_n \geq 3$ and the sequence (t_n) is increasing then convergent; finally (t_n) converge to 27, the corresponding parameter of $\mathcal{T}$. The sequences $(s_n), (t_n), (u_n)$ converge to $9, 27, 27$; therefore $(a_n), (b_n), (c_n)$, the lengths of the edges of $\mathcal{T}_n$, converge to $\sqrt{3}$. $\square$

Let ${a_n}^2 = 3 + h_n, {b_n}^2 = 3 + k_n, {c_n}^2 = 3 + l_n, \delta_n = (h_n, k_n, l_n)$. $||.||$ refers to the euclidean norm. Now we show an important estimate.

Lemma 4. $h_n + k_n + l_n \sim \dfrac{1}{3}(h_n k_n + k_n l_n + l_n h_n) \sim \dfrac{-1}{6}||\delta_n||^2$ *when n tends to ∞.*

Proof. $u = t - 4s^2 \Leftrightarrow 3(h_n + k_n + l_n) = -(h_n + k_n + l_n)^2 + (h_n k_n + k_n l_n + l_n h_n)$
$-h_n k_n l_n$. Thus $3(h_n + k_n + l_n) = -(h_n + k_n + l_n)^2 + (h_n k_n + k_n l_n + l_n h_n) + O(||\delta_n||^3) = h_n k_n + k_n l_n + l_n h_n + O(||\delta_n||^3)$.
$h_n k_n + k_n l_n + l_n h_n = \dfrac{1}{2}(h_n + k_n + l_n)^2 - \dfrac{1}{2}(h_n^2 + k_n^2 + l_n^2) \sim -\dfrac{1}{2}(h_n^2 + k_n^2 + l_n^2)$. $\square$

Proposition 13. *If G_n is the centroid of $\mathcal{T}_n$ then $OG_{n+1} \sim {OG_n}^2$ when n tends to ∞. Thus the sequence (OG_n) converges with order 2.*

Proof. ${OG_n}^2 = \dfrac{h_n + k_n + l_n}{-9}$; ${OG_{n+1}}^2 = 1 - \dfrac{s_{n+1}}{9}$. Using Maple and Lemma 1, we obtain the Taylor series of ${OG_{n+1}}^2$ with the precision $O(||\delta_n||^5)$:
${OG_{n+1}}^2 = \dfrac{N_n}{-81^2}$ where

$$N_n = 81(h_n + k_n + l_n)^2 + 18(h_n + k_n + l_n)(2{h_n}^2 + 2{k_n}^2 + 2{l_n}^2 + h_n k_n + k_n l_n + l_n h_n) + O(||\delta_n||^5)$$
$$= 27(h_n + k_n + l_n)(3(h_n + k_n + l_n) - 2(h_n k_n + k_n l_n + l_n h_n)) + O(||\delta_n||^5)$$

$\sim -81(h_n + k_n + l_n)^2$. Finally ${OG_{n+1}}^2 \sim \dfrac{(h_n + k_n + l_n)^2}{81} = {OG_n}^4$. $\square$

Remark. • *We can deduce that there exists $\lambda \in (0, 1)$, that depends upon a, b, c, such that $OG_n \sim \lambda^{2^n}$. If $\mathcal{T}_0$ is close to a flat triangle then λ is close to 1. Of course if $\mathcal{T}_0$ is close to an equilateral triangle then λ is close to 0 (see $\mathcal{T}_0$ as the result of a large number of iterations.)*
• *The result obtained in [4]: $OG_{n+1} = O({OG_n}^2)$ is weaker and does not give the preceding estimate of OG_n.*

7.5. **The main result in dimension two.**

Theorem 5. *The sequences of triangles $(\mathcal{T}_{2i})_{i \in \mathbb{N}}$ and $(\mathcal{T}_{2i+1})_{i \in \mathbb{N}}$ are well defined and converge with at least quadratic speed to two equilateral triangles that are symmetric with respect to O.*

Proof. For all sufficiently large n, G_n and G_{n+1} are close to O then close to the middle points of the segments $[A_n A_{n+1}]$ and $[A_{n+1} A_{n+2}]$; thus for all sufficiently large n, $A_n A_{n+2} \leq 3 G_n G_{n+1}$. Let $p \in \mathbb{N}^*$; $A_n A_{n+2p} \leq 3 \sum_{k=n}^{n+2p-2} G_k G_{k+1} \leq 3 \sum_{k=n}^{n+2p-2} (OG_k + OG_{k+1}) \leq 6 \sum_{k=n}^{n+2p} OG_k$. The series $\sum OG_n$ converges, then $(A_{2n})_n$ is a Cauchy sequence; thus it converges to A^∞. By the same way $(A_{2n+1})_n$ converges to A'^∞, the symmetric of A^∞ with respect to O.

Moreover $A_{2n} A^\infty \leq 6 \sum_{k=2n}^{\infty} OG_k \leq 12 \times OG_{2n}$ for all sufficiently large n; therefore $A_{2n} A^\infty = O(\lambda^{2^{2n}})$ and the sequences $(\mathcal{T}_{2i})$ and $(\mathcal{T}_{2i+1})$ converge with at least quadratic speed. $\square$

8. Conclusion

We mention some questions we can ask about these dynamical systems.

i) Can one generalize our results concerning the d-simplices for $d > 3$? We note that the complexity of the computations increases very quickly.

ii) What occurs if $\mathcal{T}_0$ is a degenerate d-simplex that admits a circumsphere in $\mathbb{R}^e$ with $e < d$? For example, $\mathcal{T}_0$ could be a convex polyhedron that admits a circumsphere in $\mathbb{R}^3$.

iii) More generally what occurs if we replace the centroid of $\mathcal{T}_i$ with some barycenter of the vertices of $\mathcal{T}_i$?

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